A Halfspace Theorem for Mean Curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

We prove a vertical halfspace theorem for surfaces with constant mean curvature $H = \frac{1}{2}$, properly immersed in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. The proof is a geometric application of the classical maximum principle for second order elliptic PDE, using the family of non compact rotational $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

1 Introduction

This is a revised version of the article that we submit before. There was a problem in the construction of graphical ends. We are presently working to fix it (replace the previous boundary with a planar boundary curve and use Perron method). The main geometric constructions will be mantained. Here we present the halfspace type theorem, that correspond to Section 4 of the previous article.
D. Hofmann e W. Meeks proved a beautiful theorem on minimal surfaces, the so-called "Halfspace Theorem" in [3]: there is no non planar, complete, minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. We focus in this paper complete surfaces with constant mean curvature $H = \frac{1}{2}$ in the product space $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane and $\mathbb{R}$ is the set of real numbers. In the context of H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, it is natural to investigate about halfspace type results.
Before stating our result we would like to emphasize that, in last years there has been work in H-surfaces in homogeneous 3-manifolds, in particular in the product space $\mathbb{H}^2 \times \mathbb{R}$: new examples were produced and many theoretical results as well.
Halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many vertically bounded complete minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [9]. On the contrary, we are able to prove the following result for $H = \frac{1}{2}$ surfaces.

Theorem 1.1 *Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a complete surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ can not be properly immersed in the mean convex side of S.*

In [4], L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for surfaces on one side of a horocylinder.
The result in [4] is different in nature from our result because in [4], the "halfspace" is one side of a horocylinder, while for us, the "halfspace" is the mean convex side of the rotational simply connected surface.
The proof of our result is a geometric application of the classical maximum principle to surfaces with constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$.

Maximum Principle. *Let S_1 and S_2 be two connected surfaces of constant mean curvature $H = \frac{1}{2}$. Let $p \in S_1 \cap S_2$ be a point such that S_1 and S_2 are tangent at p, the mean curvature vectors of S_1 and S_2 at p point towards the same side and S_1 is on one side of S_2 in a neighborhood of p. Then S_1 coincide with S_2 around p. By analytic continuation, they coincide everywhere.*

The proof of the Maximum Principle is based on the fact that a constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ locally satisfies a second order elliptic PDE (cf. [2], [1] where the author prove the Maximum Principle in $\mathbb{R}^n$; the proof generalizes to space forms and to $\mathbb{H}^2 \times \mathbb{R}$ as well).
We notice that our surfaces are not compact, while the classical maximum principle applies at a finite point. It will be clear in the proof of Theorem 1.1 that we that we are able to reduce the analysis to finite tangent points, because of the geometry of rotational surfaces of constant mean curvature $H = \frac{1}{2}$.
Our halfspace Theorem leads to the following conjecture (strong halfspace theorem).

Conjecture. *Let Σ_1, Σ_2 be two complete properly embedded surfaces with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected one. Then Σ_i can not lie in the mean convex side of Σ_j, $i \neq j$.*

For $H > \frac{1}{\sqrt{2}}$ the conjecture is true and it is known as maximum principle at infinity (cf. [6]).

2 Vertical Halfspace Theorem

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ in [8]. A careful description of the geometry of these surfaces is contained in the Appendix of [7].
Here we recall some properties of rotational surfaces of constant mean curvature $H = \frac{1}{2}$.
For any $\alpha \in \mathbb{R}_+$, there exists a rotational surface $\mathcal{H}_\alpha$ of constant mean curvature $H = \frac{1}{2}$.
For $\alpha \neq 1$, the surface $\mathcal{H}_\alpha$ has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are graphs over the exterior of a disk D_α of hyperbolic radius $r_\alpha = |\ln \alpha|$.

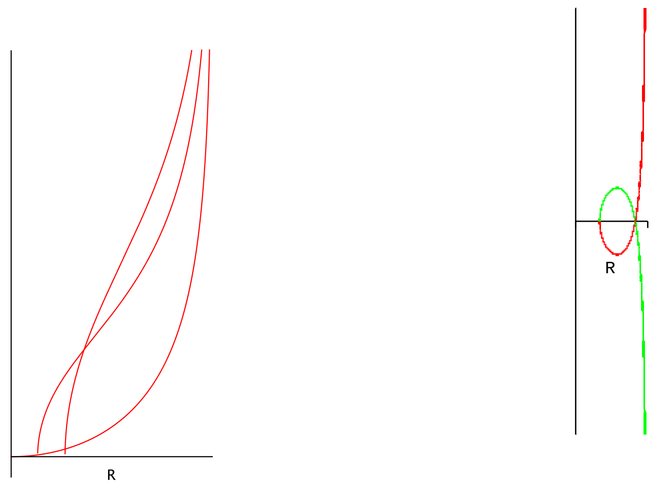

Figure 1: $H = \frac{1}{2}$: the profile curve in the embedded and immersed case ($R = \tanh \rho$).

By graph we mean the following: the graph of a function u defined on a subset Ω of $\mathbb{H}^2$ is $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$. When the graph has constant mean curvature H, u satisfies the following second order elliptic PDE

$$\mathrm{div}_{\mathbb{H}} \left(\frac{\nabla_{\mathbb{H}} u}{W_u} \right) = 2H \tag{1}$$

where $\mathrm{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2 \times \{0\}$.
Furthermore, up to vertical translation, one can assume that $\mathcal{H}_\alpha$ is symmetric with respect to the horizontal plane $t = 0$. For $\alpha = 1$, the surface $\mathcal{H}_1$ has only one end and it is a graph over $\mathbb{H}^2$ and it is denoted by S.
When $\alpha > 1$ the surface $\mathcal{H}_\alpha$ is not embedded. The self intersection set is a horizontal circle on the plane $t = 0$. For $\alpha < 1$ the surface $\mathcal{H}_\alpha$ is embedded.
For any $\alpha \in \mathbb{R}_+$, each end of the surface $\mathcal{H}_\alpha$ is the vertical graph of a function u_α over the exterior of a disk D_α of radius r_α. The asymptotic behavior of u_α has the following form: $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}}$, $\rho \longrightarrow \infty$, where ρ is the hyperbolic distance from the origin. The positive number $\frac{1}{\sqrt{\alpha}} \in \mathbb{R}_+$ is called the *growth* of the end.
The function u_α is vertical along the boundary of D_α. Furthermore the radius r_α is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \longrightarrow 0$ or $\alpha \longrightarrow \infty$. As we pointed out before, the function $u_1 = 2 \cosh \left(\frac{\rho}{2} \right)$ is entire and its graph corresponds to the unique simply connected example S.
Notice that, any end of an immersed rotational surface ($\alpha > 1$) has growth smaller than the growth of S, while any end of an embedded rotational surface ($\alpha < 1$) has growth greater than the growth of S.

Theorem 1.1 is called "vertical" because the end of the surface Σ is vertical, as it is contained in the mean convex side of S.

Proof of Theorem 1.1. One can assume that the surface S is tangent to the slice $t = 0$ at the origin and it is contained in $\{t \geq 0\}$. Suppose, by contradiction, that Σ is contained in the mean convex side of S. Lift vertically S. If there is an interior contact point between Σ and the translation of S, one has a contradiction by the maximum principle. As Σ is properly immersed, Σ is asymptotic at infinity to a vertical translation of S. One can assume that the surface Σ is asymptotic to the S tangent to the slice $t = 0$ at the origin and contained in $\{t \geq 0\}$.
Let h be the height of one lowest point of Σ. Denote by $S(h)$ the vertical lifting of S of ratio h. One has one of the following facts.

- $S(h)$ and Σ has a first finite contact point p : this means that $S(h-\varepsilon)$ does not meet Σ at a finite point, for $\varepsilon > 0$ and then $S(h)$ and Σ are tangent at p with mean curvature vector pointing in the same direction. In this case, by the maximum principle $S(h)$ and Σ should coincide. Contradiction.

- $S(h)$ and Σ meet at a point p, but p is not a first contact point.Then, for ε small enough, $S(h-\varepsilon)$ intersect Σ transversally.

Denote by W the non compact subset of $\mathbb{H}^2 \times \mathbb{R}$ above S and below $S(h-\varepsilon)$.
It follows from the maximum principle that there are no compact component of Σ contained in W. Denote by Σ_1 a non compact connected component of Σ contained in W. Note that the boundary of Σ_1 is contained in $S(h-\varepsilon)$. Consider the family of rotational non embedded surfaces $\mathcal{H}_\alpha$, $\alpha > 1$. Translate each $\mathcal{H}_\alpha$ vertically in order to have the waist on the plane $t = h - \varepsilon$. By abuse of notation, we continue to call the translation, $\mathcal{H}_\alpha$. The surface $\mathcal{H}_\alpha$ intersects the plane $t = h - \varepsilon$ in two circles. Denote by ρ_α the radius of the larger circle. Denote by $\mathcal{H}_\alpha^+$, the part of the surface outside the cylinder of radius ρ_α. Notice that $\mathcal{H}_\alpha^+$ is embedded. By the geometry of the $\mathcal{H}_\alpha^+$, when α is great enough, say α_0, $\mathcal{H}_{\alpha_0}^+$ is outside the mean convex side of S. Then, $\mathcal{H}_{\alpha_0}^+$ does not intersect Σ. Furthermore, when $\alpha \longrightarrow 1$, $\mathcal{H}_\alpha^+$ converge to $S(h-\varepsilon)$. Now, start to decrease α from α_0 to one. Before reaching $\alpha = 1$, the surface $\mathcal{H}_\alpha^+$ first meets S and then touches Σ_1 tangentially at an interior finite point, with Σ_1 above $\mathcal{H}_\alpha^+$. This depends on the following two facts.

- The boundary of Σ_1 lies on $S(h-\varepsilon)$ and the boundary of any of the $\mathcal{H}_\alpha^+$ lies on the horizontal plane $t = h - \varepsilon$.

- The growth of any of the $\mathcal{H}_\alpha^+$ is strictly smaller than the growth of S. Thus the end of $\mathcal{H}_\alpha^+$ is outside the end of S.

The existence of an such interior tangency point is a contradiction by the maximum principle.

$\square$

References

[1] D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order,* Springer- Verlag (1998).

[2] H. Hopf: *Differential Geoemtry in the Large,* Lecture Notes in Mathematics, 1000, Springer-Verlag (1983).

[3] D. Hoffman, W. Meeks III: *The Strong Halfspace Theorem for Minimal Surfaces,* Inven. Math. 101, n.1 (1990) 373-377.

[4] L. Hauswirth, H. Rosenberg, J. Spruck: *On Complete Mean Curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$,* http://www.institut.math.jussieu.fr∼rosen//hrs11.pdf

[5] B. Nelli, H. Rosenberg: *Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. Braz. Math. Soc. 33 (2002) 263-292; Errata Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$, [Bull. Braz. Math. Soc., New Series 33 (2002), 263-292] Bull. Braz. Math. Soc., New Series 38(4) (2007) 1-4.

[6] B. Nelli, H. Rosenberg: *Global Properties of Constant Mean Curvature Surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Pacific Journ. Math. 226, No 1 (2006) 137-152.

[7] B. Nelli, R. Sa Earp, W. Santos, E. Toubiana: *Existence and Uniqueness of H-surfaces with one or two parallel convex curves as boundary in $\mathbb{H}^2 \times \mathbb{R}$*, Annals of Global Analysis and Geometry, DOI 10.1007/s10455-007-9087-3, **33**, N. 4 (2008) 307-321.

[8] R. Sa Earp, E. Toubiana: *Screw Motion Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Illinois Jour. of Math. 49 n.4 (2005) 1323-1362.

[9] R. Sa Earp, E. Toubiana: *An asymptotic theorem for minimal surfaces and existence results for minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$*, published on line May 2008, Mathematische Annalen, DOI 10.1007/s00208-008-0237-0, 342, N. 2 (2008) 309-331

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